On Commutativity of *-Prime Rings with Generalized (α, β)-Derivations

M. K. ABU NAWAS and R. M. AL-OMARY

1Faculty of Science, Northern Border University, Rafha, P. O. Box 840, Saudi Arabia.
2Department of Mathematics, Faculty of Education,Ibb University, Al-Naderah, Yemen.

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ABSTRACT

Let *(R, *)* be a 2-torsion free *-prime ring with involution *. An additive mapping *F:* $R \rightarrow R$ is called a generalized (α, β)- derivation on *R* if there exists an (α, β)- derivation *d:* $R \rightarrow R$ such that the equation $F(xy) = F(x)\alpha(y)+\beta(x)d(y)$ holds for all *x, y* in *R*. In this paper we have investigated the commutativity of *R* such that *R* is a *-prime ring admitting generalized (α, β)- derivations *(F, d)* and *(G, g)* that satisfying certain properties for all $x, y \in R$.

Key words: Prime rings, (α, β)-Derivations, Generalized (α, β)-Derivations, *-prime rings, Commutativity of rings.

INTRODUCTION

In the last four decades, several authors discussed the commutativity of the prime rings and the semiprime rings that admitting automorphisms, derivations or generalized derivations that are centralizing or commuting on an appropriate subset of *R* (see [1], [2], [3], [4], [8] and [10]). Related concepts such as S -prime rings and Lie structure of prime rings with generalized (a , b)-derivations are also discussed (see for example: [5], [6], [7] and [9]).

Let *R* be an associative ring with center *Z* (R) and involution $*$. For each $x, y \in R$, the symbol $[x, y]$ represents the commutator $xy - yx$ and the symbol $x \circ y$ stands for the skew-commutator $xy + yx$. An additive mapping $x \mapsto x^*$ on a ring R is called an involution if $(x^*)^* = x$ and $(xy)^* = y^* x^*$ for all . A ring equipped with an involution is said to be a -prime ring if $ab = ab^* = \{0\}$ implies $a = 0$ or $b = 0$ for any *a, b* . Obviously, every prime ring equipped with involution is -prime. The converse, in general, is not true. The set of all symmetric and skew-symmetric elements of a - ring will be denoted by. An additive mapping is called a derivation if for all. In particular, for a fixed , the mapping given by is a derivation called an inner derivation. Let and be endomorphisms of *R*. An additive map is called an -derivation if for all . A (1, 1)-derivation is called simply a derivation, where 1 is the identity map on *R*. For a fixed , the map $d_a : R\ \ R$ given by $d_a(x) = [a, x]$ for all *x R* is an -derivation called an -inner derivation. An additive mapping *F: R R* is called a generalized -inner derivation if $F(x) = a\ (x) +\ (x)b$, for some fixed *a, bR* and for all *x R*. A simple computation yields that if *F* is a generalized inner derivation, then for all *x, y R*, we have $F(x y) = F(x)\ (y) +\ (x)\, d_{b}\ (y)$, where d_{b} is an -inner derivation. An additive map *F : R R* is called a generalized derivation associated with an -derivation *d : R R* if $F(x y) = F(x)\ (y) +\ (x)d(y)$ *for all x, y R,* and is denoted by *(F, d)*.

In this paper, we have investigated the commutativity of R such that R is a -prime ring admitting generalized -derivations *(F, d)* and *(G, g)* that satisfying certain properties for all *x, y R*. Throughout this paper, the ordered pair *(F, d)* stands for a generalized - derivation *F* associated with an -derivation *d*.

Preliminaries

We shall use the following well known basic identities that hold for any *x, y, z R* and any automorphisms α , α on *R*:

- $[x\,y, z]_{\alpha,\beta} = x[y, z]_{\alpha,\beta} + [x, \beta(z)]y = x[y, \alpha(z)] + [x, z]_{\alpha\beta}y$;
- $[x, y\,z]_{\alpha\beta} = \beta(y)[x, z]_{\alpha\beta} + [x, y]_{\alpha\beta}\alpha(z)$;
- $(x^\circ (y\,z)_{\alpha\beta}) = (x^\circ y)_{\alpha\beta}\alpha(z)\text{-}\beta(y)[x, z]_{\alpha\beta} = \beta(y)(x^\circ z)_{\alpha\beta} + [x, y]_{\alpha\beta}\alpha(z)$;
- $((x\,y)\,z)_{\alpha\beta} = x(y^\circ z)_{\alpha\beta} - [x, \beta(z)]y = (x\,^\circ z)_{\alpha\beta}y + x[y, \alpha(z)]$.

It is known that for a prime ring R and a nonzero element *aZ(R),* if *a bZ(R)*, then *bZ(R)*. The following lemma is essential to prove our results:

Lemma

Let R be a 2-torsion free semiprime ring. If α:R→R is an automorphism on R, then for any 0=zε Z(R), we have $\alpha(z)\varepsilon$ Z(R).

Proof

We have 0=zZ(R), that is [z, r] = 0 for all rR and hence ([z, r]) = 0. Since is an automorphism on R, we have [α(z), α(r)] = 0. Now, replacing r by α^{-1} (s), we get [α(z), s] = 0 for all s εR. Therefore, α(z) εZ(R) as required. Ž

Theorem

[4 : Theorem 3.5] *Let R be a 2-torsion free prime ring and U a nonzero square-closed Lie ideal of R. Suppose that R admits a generalized (*α, β*)-derivation (F, d) such that F(u°v) = 0 for all u, v* ε*U. If d = 0, then UZ(R).*

RESULTS

Theorem

Let R be a *-prime ring and , α, β be automorphisms on R. Suppose that R admits generalized (α, β)-derivations (F, d) and (G, g) such that {0} g(Z(R)) Z(R) and [F(x),G(y)] [x, y] Z(R), for all x, y R. If F = 0 (respectively G = 0) or d 0 (respectively g 0), then R is commutative where d and g are -derivations.

Proof

Let *F* and *G* be generalized -derivations on *R*. If *F = 0 (*or *G = 0)*, then for any *x, y R*, we have

[x, y] ε Z(R) ...(3.1)

Replacing *x* by *x (y)* in (3.1) gives *[x, y] (y) Z(R)*, which implies, for all *x, y, c R,*

[[x, y] (y), c] = [x, y] [(y), c] = 0.

Again if we replace c by c(m) and use the last expression we get

$[x, y]_{\alpha,\beta}$ R[α(y), α(m)] = {0}, for all x, y, m R ...(3.2)

For any *y* ε *S(R) R*, we have

$[x, y]_{\alpha,\beta}$R[(y), (m)] = $[x, y]_{\alpha,\beta}$R[(y), (m)]* = 0.

Thus, for any y ε S(R) R, the *-primeness of *R* yields that either $[x, y]_{\alpha,\beta} = 0$ or *[* α*(y),* α*(m)] = 0*. Using the fact that *y - y S*(R) R*, we get $[x, y\text{-}y]_{\alpha,\beta} = 0$ or α*([y -y*,m]) = 0*, for any *x, y, m R*. If $[x, y\text{-}y]_{\alpha,\beta}= 0$, then from equation (3.2), for all *x, y, m R*, we obtain *[x, y]* ε*R[(y), (m)] = [x, y]R[(y), (m)]*. If *([y * y,m]) = 0*, then from (3.2), we obtain

$[x, y]_{\alpha,\beta}$R[α (y), α(m)] = $[x, y]_{\alpha,\beta}$R[(y), α (m)].

Consequently, for all *yR* either $[x, y]_{\alpha,\beta} = 0$ or *[* α*(y), (m)] = 0*. Now, let *A ={y* ε*R / [y, m] = 0}* and $B = \{y\ \varepsilon R\ /\ [x, y]_{\alpha,\beta} = 0\}$. Then *A, B* are both additive subgroups of *R* and *R = A B*. But a group cannot be a union of its two proper subgroups and hence either *R = A* or *R = B*. If *A = R* then we have *[y, m] = 0* for all *y, m R* and hence *R* is commutative. If *B = R*, then we have

$[x, y]_{\alpha,\beta} = 0$, for all x, y ε R ...(3.3)

Replacing *x* by *xc* in (3.3) gives *[c,* β*(y)]x = 0*, for all *x, c, y R*, that is *[c, (y)]Rx = 0*. For any *x S(R) R*, we have *[c, (y)]Rx = 0 = [c, (y)]Rx*. Thus, the *- primeness of *R* yields that *[c, (y)] = 0* for all *c, y R*. As is an automorphism on *R*, we obtain that *R* is commutative.

Assume that *g = 0 (or d = 0)*. Then for all *x, y* ε*R*, we have

$[F(x),G(y)] - [x, y]\ \ Z(R)$ ...(3.4)

Since *{0} =g(Z(R)) Z(R)*, for any $z\ \varepsilon\ Z(R)$ replacing *y* by *y z*, in (3.4) and using Lemma 2.1 give

$[F(x),\ (y)]g(z) -\beta\ (y)[x, z]_{\alpha,\beta}\ \ \varepsilon Z(R)$ ...(3.5)

Again, replacing *y* by *my* in (3.5) gives
$\beta(m)\{[F(x),\ \beta(y)]g(z)\ -\beta\ \ (y)[x,\ z]_{\alpha,\beta}\} + [F(x),\ \beta(m)]\ \beta(y)g(z)\varepsilon\ \ Z(R)$,

or equivalently,

$[\ \beta(m)\{[F(x),\ \ \beta(y)]g(z)\ -\beta\ (y)[x, z]_{\alpha,\beta}\ \} + [F(x),\ \ \beta(m)]\ \beta(y)g\varepsilon(z),\ \beta(m)] = 0$,

which implies $[[F(x), \beta(m)]\ \beta(y)g(z),\ \beta(m)] = 0$ for all $x, y, m\ \varepsilon\ R$. Since *{0} g(Z(R)) Z(R)* and *R* is *- prime, we get

$[F(x),\ \beta(m)][\ \beta(y),\ \beta(m)] + [[F(x),\ \beta(m)],\ \ \beta(m)]\ \beta(y) = 0$.

Now, replacing *y* by *y x* in the last expression gives $[F(x), \beta(m)]\ \ \ \beta(y)[\ \beta(x),\ \beta(m)] = 0$, which implies

$[F(x),\ \ \beta(m)]R[\ \beta(x),\ \ \beta(m)] = \{0\}$ for all $x, m\ \varepsilon R$, ...(3.6)

as β is an automorphism on *R*. Next, for all $x\ \varepsilon R$, let $x\ \varepsilon\ S(R)\ R$, then equation (3.6) yields $[F(x), \beta(m)]R[\ (x),\ \beta(m)] = 0 = [F(x),\ \beta(m)]R([\ (x),\ \beta(m)])$. Hence, the *-primeness of *R* implies that either $[F(x), \beta\ (m)] = 0$ or $[x, m] = 0$. For any $x\ \ R$, we have $x\text{-}x^*\ \ S(R)\ R$. Thus, for some $x\ \ R$, $[F(x\text{-}x^*),\ \ (m)] = 0$ or $[x - x, m] = 0$. If $[F(x - x),\ \ (m)] = 0$, then for all $x, m\ \ R$ we have $[F(x),\ \ \beta(m)]R[\ (x),\ \ \beta(m)] = ([F(x),\ \ \beta(m)])\ R[\ (x),\ \ \beta(m)]$, which means that either $[F(x),\ \ \beta(m)] = 0$ or $[x ,m] = 0$. If $[x - x,m] = 0$, then equation (3.6) yields $[F(x),\ \ (m)]R[\ \beta(x),\ \ \beta(m)] = [F(x),\ \ \beta(m)]R([(x),\ \beta(m)])$ which means that either $[F(x),\ \ \beta(m)] = 0$ or $[x ,m] = 0$. Now, we let $A = \{x\ \ R\ /\ [x, m] = 0\}$ and $B = \{x\ \ R\ /\ [F(x),\ \ (m)] = 0\}$. Then *A* and *B* are both additive subgroups of *R* whose union is *R*. Using Brauer's trick we have either $A = R$ or $B = R$. If $A = R$, then we have $[x, m] = 0$ for all $x, m\ \ R$ which means that R is commutative. If $B = R$ then $[F(x),\ \ \beta(m)] = 0$ for all $x, m\ \ R$. Since is an automorphism on *R*, we get $F(x)\ \ Z(R)$ for all $x\ \ R$. Thus, our hypothesis in (3.4), yields $[x, y]_{\alpha,\beta}\ \ Z(R)$. A similar argument as that after (3.1) assures that *R* is commutative. Therefore, for each case, we have *R* is commutative as required.

Theorem

Let R be a *- prime ring and, be automorphisms on R. Suppose that R admits a generalized (α,β)-derivation (F, d) satisfying any of the following conditions for all x, y ε R:

(i) $[F(x), x]_{\alpha,\beta} = [y, x]_{\alpha,\beta}$,
(ii) $(F(x)\ {}^\circ x)_{\alpha,\beta} = (y\ \ x)_{\alpha,\beta}$,
(iii) $(F(x)\ \ {}^\circ x)_{\alpha,\beta} = [y, x]_{\alpha,\beta}$.

If F = 0 (or d 0), then R is commutative.

Proof

If $F = 0$, then $[y, x]_{\alpha,\beta} = 0$ for all $x, y\ \varepsilon\ R$. Thus, using similar arguments as that after equation (3.3) gives that *R* is commutative. Henceforth, we shall assume that $d =0$. So, for all $x, y\ \varepsilon\ R$, we have

$[F(x), x]_{\alpha,\beta} = [y, x]_{\alpha,\beta}$ ...(3.7)

Replacing *x* by *x + m* in (3.7) gives

$[F(x),m]_{\alpha,\beta}\ + [F(m), x]_{\alpha,\beta}\ = 0$, for all $x, m\varepsilon R$ ...(3.8)

Now, we again replace *m* by *mx* in (3.8) to get
$\beta([m, x])d(x) + \beta(m)[d(x), x]\alpha,\beta\ + \beta(m)[F(x), x]\ \ \alpha, \beta = 0$ ...(3.9)

Also, replacing *m* by *w m* in (3.9) gives *([w, x]) (m)d(x) = 0*. Since is an automorphism on *R*, we get
$\beta([w, x])Rd(x) = \{0\}$ for all $w, x\ \varepsilon R$ ...(3.10)

If $x\varepsilon S\ (R)\ R$, then $([w, x])Rd(x) = \beta([w, x])\ Rd(x)$. Thus, for some $x\ \varepsilon\ S\ (R)\ \ R$, the *- primeness of *R* yields either $\beta([w, x]) = 0$ or $d(x) = 0$. But for any $x\ \varepsilon\ R$, $x - x\ \ S_*\ (R)\ R$. Thus, for some $x\ \ \varepsilon\ R$, either $[w, x\ \text{-}x] = 0$ or $d(x\ \text{-}x) = 0$. If $[w, x - x] = 0$, then equation (3.10) follows that $([w, x])Rd(x) = 0 =\ \ ([w, x])\ Rd(x)$. Hence the *- primeness of *R* yields either $([w, x]) = 0$ or $d(x) = 0$. If $d(x\text{-}x) = 0$ then $d(x) = (d(x))$ for all $x\ \ R$. Consequently, for all $x, w\ R$, either $([w, x]) = 0$ or $d(x) = 0$. Let $A = \{x\ \ R\ /\ d(x) = 0\}$ and $B = \{x\ \ R\ /\ [w, x] = 0\}$. Then *A* and *B* are both additive subgroups of *R* whose union is *R*. Using Brauer's trick we have either $A = R$ or $B = R$. If $A = R$ then $d(x) = 0$ for all *x*

R, a contradiction. If *B = R*, then *[w, x] = 0* for all *x, w R* and hence *R* is commutative.

If *F = 0* then for all *x, y R*, we have

$(x°y)_{\alpha,\beta} = 0$ ...(3.11)

Replacing *y* by *y m* in (3.11), we get β*(y)[x, m]*$_{\alpha,\beta}$ *= 0* for all *x, m* ε *R*. Again, for any *z* ε *R*, replace *y* by *y* ε *z* in the last expression, to get *y* $^{\beta-1}$*R([x, m]) = 0 = yR* $^{\beta-1}$ *([x, m])*. Applying the -primeness of R, we get *([x, m]*$_{\alpha,\beta}$ *) = 0* for all *x, m R*. Since is an automorphism on *R*, we obtain *[x ,m] = 0* for all *x, m R*. Now, using similar arguments as that follow equation (3.3), we get the required result. Therefore, we shall assume that *d=0* . Thus, for any *x, y R* we have

(F(x) x)= (y x) ...(3.12)

Replacing *x* by *x + m* for any *m* ε *R* in (3.12) gives

$(F(x) ° m)_{\alpha,\beta} + (F(m) ° x)_{\alpha,\beta} = 0$ ...(3.13)

Again we replace *m* by *mx* in (3.13) to get

$-\beta(m)[F(x), x]_{\alpha,\beta} + \beta(m)(d(x) °x) - [\beta(m), \beta(x)]d(x) = 0.$ ...(3.14)

Now, replacing m by *w m* in (3.14) gives *([w, x]) (w)d(x) = 0* for all *x, w R*. Since is automorphism on R, we get

$\beta([w, x])Rd(x) = \{0\}$ ...(3.15)

This is the same as equation (3.10), hence continuing in the same manner as above gives that R is commutative.

If *F = 0* then, we have *[x, y]* $_{\alpha,\beta}$ *= 0* for all *x, y*ε *R*. Hence we use similar arguments as that after (3.3) to get the required result. Therefore, we shall assume that *d=0*. So, for any x, y ε R, we have

$(F(x) °x)_{\alpha,\beta} = [y, x]_{\alpha,\beta}$ ...(3.16)

For any *m* in *R*, replacing *x* by *x + m* in (3.16) gives

$(F(x)°m)_{\alpha,\beta} + (F(m) °x)_{\alpha,\beta} = 0$ ...(3.17)

Applying the same arguments that follow equation (3.13) yields the required result. Ž

The following corollaries are immediate consequences of theorems 3.1 and 3.2 or by using the same techniques taking in our account theorem 2.1 for corollary 3.4:

Corollary

Let R be a 2-torsion free *-prime ring and α,β be automorphisms on R. Suppose that R admits a generalized (α,β)-derivation (F, d) such that

$$F([x, y]) - [F(x), y]_{\alpha,\beta} = [d(y), x]_{\alpha,\beta}$$

for all x, y ε R. If F = 0 (or d = 0), then R is commutative.

Corollary

Let R be a 2-torsion free *-prime ring and α,β be automorphisms on R. Suppose that R admits a generalized (α,β) -derivation (F, d) such that

$F([x, y]) - (F(x) ° y) = [d(y), x]_{\alpha,\beta}$

for all x, y R. If F = 0 (or d 0), then R is commutative.

Corollary

Let R be a *-prime ring and α,β be automorphisms on R. Suppose that R admits generalized (α,β) -derivations (F, d) and (G, g) such that

F([x, y]) = [(y),G(x)],

for all x, yεR. If F = 0 (respectively G = 0) or d=0 (respectively g=0), then R is commutative.

Corollary

Let R be a *-prime ring and α,β be automorphisms on R. Suppose that R admits generalized (α,β) -derivations (F, d) and (G, g) such that

$$F(x°y) = ((y)°G(x)),$$

for all x, yεR. If F = 0 (respectively G = 0) or d=0 (respectively g=0), then R is commutative.

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